Synopsis of the Ph.D. Thesis entitled

A generalized study on Graceful labeling of Graphs

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Publications

1. K.Kayathri and R.Amutha, Edge-graceful labeling of connected graphs, Electronic Notes in Discrete Mathematics, Elsevier, 53(2016), 287-296.

2. R.Amutha and K.Kayathri, Enumeration of edge-graceful labelings of K_3, K_4, K_5, Mathematical Sciences International Research Journal : Volume 5 Issue 2 (2016), 21- 26. ISSN 2278-8697, ISBN 978-93-84124-93-9.

3. K.Kayathri and R.Amutha, Directed Edge-Graceful Labelings, (Communicated).

Presentations

1. Presented a paper entitled "Edge-Graceful Labelings of Connected graphs" in the International Conference on Graph Teory and its Applications held at Amrita School of Engineering, Coimbatore, during 16-19 December 2015.

2. Presented a paper entitled "Enumeration of Edge-Graceful Labelings of K_3, K_4, K_5" in the International Conference on Mathematics and Computer science held Nirmala College, Coimbatore, during 15-17, December 2016.

3. Presented a paper entitled "$K-$edge gracefull labeling" in the National Conference on Mathematical Modeling Bioresource Management held at Thiagarajar College, Madurai, during 6-7, April 2017.

Synopsis

A generalized study on Graceful labeling of Graphs

Graph Theory is one of the most fascinating and vibrant area of Mathematics. Some of the interesting fields in graph theory are enumeration of graphs, domination in graphs, algorithmic graph theory, topological graph theory, fuzzy graph theory, labeling of graph theory, etc. As graph labeling is one of the major research area in graph theory, we focus on graph labelings. Labeled graphs are becoming an increasingly useful family of mathematical models for broad and wide range of applications. Graph labeling is applied in data mining, image processing, software testing, information security, communication networks, etc. In this thesis, we deal with enumeration of graph labelings.

Most graph labeling methods trace their origin from the concept of β- valuation, introduced by Alex Rosa in 1967 [14]. He introduced it to attack the problem of cyclically decomposing the complete graph into other graphs.

Independently, Golomb[10] studied the same type of labeling and called this labeling graceful labeling. There are various types of graph labeling such as graceful labeling, harmonious labeling, cordial labeling, arithmetic labeling, Skolem graceful labeling, magic labeling, antimagic labeling, α-labeling, prime labeling, mean labeling and vertex graceful labeling, which are investigated by sevaral authors. A dynamic survey of graph labeling is carried out by Gallian[8].

Apart from the theoretical developments, applications of graph labeling have been found in X-ray crystallography, coding theory, radar, circuit design, communication design, missile guidance and database management. Particularly interesting applications of graph labeling can be found in Bloom and Golomb's papers [4, 5].

Edge-graceful labeling, a dual concept of graceful labeling was introduced by Lo [15] in 1985. A graph $G(V, E)$ is said to be *edge-graceful* if there exists a bijection f from E to $\{1, 2, ..., |E|\}$ such that the induced mapping f' from

V to $\{0, 1, 2, ..., |V| - 1\}$ given by $f'(x) = \sum f(xy) \pmod{|V|}$ taken over all edges incident with x, is a bijection.

Lo [15] proved that all odd cycles are edge-graceful. A necessary condition for a graph with p vertices and q edges to be edge-graceful is that $q(q+1) \equiv \frac{p(p-1)}{2} \pmod{p}$ [15].

Lee [16] conjectured that any connected simple (p, q) graph with $q(q+1) \equiv \frac{p(p-1)}{2} \pmod{p}$ is edge-graceful. Lee and Murthy [11] proved that K_n is edge-graceful if and only if $n \not\equiv 2 \pmod{4}$. Lee [17] also conjectured that all trees of odd order are edge-graceful.

The concept of k-edge-graceful labellings was first introduced in 2004 by Lee et al. [18].

A graph with p vertices and q edges is said to be *k-edge-graceful* if its edges can be labeled with $k, k+1, k+2, ..., k+q-1$, such that the vertex sums are distinct modulo p.

Note that, 1-edge-gracefulness of a graph G is equivalent to the edge-gracefulness of G. They also gave the necessary condition for the k-edge-graceful graph, a generalization of Lo's condition for edge-graceful graphs.

If a (p, q)-graph is k-edge-graceful then it satisfies the condition $q(q + 2k - 1) \equiv \frac{p(p-1)}{2} \pmod{p}$. Lee[16] defined the edge-graceful spectrum of a graph G as the set of all nonnegative integer k such that G has a k-edge-graceful labeling. In the literature, there are very few works on labelings on digraphs than there are on undirected graphs. In 1985, Bloom and Hsu [6] defined graceful labelings on directed graphs. In 2008, Bloom et al [7] defined magic labelings in directed graphs.

In her Ph.D. thesis, Vanitha [21] defined a (p, q) *graph* G to be *directed edge-graceful* if there exists an orientation of G and a labeling of the arcs of G with $\{1, 2, ..., q\}$ such that the induced mapping g on V defined by $g(v) \equiv |f^+(v) - f^-(v))| \pmod{p}$ is a bijection, where $f^+(v)$ is the sum of the labels of all arcs with head v and $f^-(v)$ is the sum of the labels of all arcs with tail v. She proved that a necessary condition for a graph with p vertices to be directed edge-graceful is that p is odd. She proved that odd paths, odd cycles, fan graphs $F_{2n}(n \geq 2)$, wheels W_{2n} are directed edge-graceful graphs.

R.Thamizharasi and R.Rajeswari [19] defined the edge-graceful labeling and k-edge-graceful labelings of digraphs. A *digraph* is said to be *edge-graceful* if there exists a bijection $f : A \to \{1, 2, ..., q\}$ such that the induced mapping $f' : V \to \{0, 1, 2, ..., p-1\}$, given by
$f'(v_i) = \sum\limits_{a \in N^-(v)} f(a)$ (mod p) is a bijection.

A *digraph* with p vertices and q edges is k*-edge-graceful* if there exists a bijection $f : A \to \{k, k+1, k+2, ..., k+q-1\}$ such that the induced mapping $f' : V \to \{0, 1, 2, ..., p-1\}$ given by $f'(v_i) = \sum\limits_{a \in N^-(v)} f(a)$ is a bijection.

In the literature, there are very few works on enumeration in labelings. D.P.Mehendale [12] developed an algorithm to generate all the gracefully labeled spanning trees in a complete graph and modified to generate all graceful spanning trees. He also obtained an upper bound on the count of gracefully labeled trees in a complete graph. Barrientos et al. [3] determined the number of graceful graphs of size n and order m. Aldred et al. [2] proved that the number of graceful labelings of P_n grows at least as fast as $\Omega(\frac{5}{3})^n$. Michal Adamaszek [1] improved this asympototic bound to $\Omega(2.37^n)$.

Not much work was done on construction of new graphs using graph labelings. Ngurh,Basloro, Tomescu[13] gave methods for construction of new super edge-magic total graphs from old ones by adding some new pendent edges. Gee-Choon Lau et al. [9] constructed new super graceful graph, by deleting an edge and adding a new edge from super graceful graph under some conditions. They also obtained new super graceful graphs by attaching k pendent edges to a vertex of a super graceful graphs.

In this thesis, we concentrate our work on construction and enumeration of new labelings from the existing labeling of graphs and digraphs. Our study includes edge-graceful labeling, directed edge-graceful labeling, arc-graceful labeling and k-arc-graceful labeling.

In the literature, there are various type of labelings. Almost all the results focus mainly on finding the labelings for some general classes of graphs. But there is no general technique to generate such labelings. Motivated by this fact we initiate our research in this direction.

In spite of the fact that edge-graceful labelings appeared almost 30 years ago, not many general techniques are known in order to generate edge-graceful labelings of graphs. Many researchers concentrate only on creating the (new) families of edge-graceful graphs. As there are very few works on generating new labeling from the existing one, we focus our work to generate new edge-graceful labelings.

Given an edge-graceful labeling f of G, we derive new edge-graceful labelings, in which only the edge labels vary, but the vertex labels remain the same. Also we enumerate these new edge-graceful labelings. Using the same technique, we derive new directed edge-graceful labelings from a given directed edge-graceful labeling a graph G. Since directed edge-graceful labelings are defined with respect to an orientation, we derive a large number of directed edge-graceful labelings by inducing new orientations. We enumerate these labelings also.

We also construct new graphs with directed edge-graceful labelings, using directed edge-graceful labeling of K_n.

As there are very few works on labelings on digraphs, we introduce new labelings called *arc-graceful labeling* and *k-arc-graceful labeling* for digraphs and extend our work for these labelings also. We divide the entire thesis into six chapters:

- Introduction
- Edge-Graceful Labelings
- Enumeration of edge-graceful labelings of K_3, K_4, K_5
- Directed Edge-Graceful Labelings
- Construction of new graphs with directed edge-graceful labeling

- Arc-graceful labelings and k-arc-graceful labelings of Digraphs

The first chapter contains all the necessary preliminary concepts, basic definitions, results that are needed for the subsequent chapters.

In Chapter 2, we derive new edge-graceful labelings from a known edge-graceful labeling and enumerate the number of new labelings derived from a given one. We also prove similar results for k-edge-graceful labelings. We consolidate our results in the following three main theorems.

Theorem 1. Let f be an edge-graceful labeling of a connected (p, q) graph G with $q = kp+r$, where $0 \leq r < p$ and k is an integer with $k \geq 1$. Let S, S' be the set of all permutations defined on the sets $\{0, 1, 2, ..., k-1\}$ *and* $\{0, 1, 2, ..., k\}$ respectively. For any p-tuple $(\pi_1, \pi_2, ..., \pi_p)$, where $\pi_1, \pi_2, ..., \pi_r \in S'$ *and* $\pi_{r+1}, \pi_{r+2}, ..., \pi_p \in S$, define a labeling $F_{(\pi_1,\pi_2,...,\pi_p)}$ such that

$$F_{(\pi_1,\pi_2,...,\pi_p)}(e) = f(e) + (\pi_i(l) - l)p \ if \ f(e) = lp + i,$$

where $0 \leq l \leq k$ if $1 \leq i \leq r$; and $0 \leq l \leq k-1$ if $r+1 \leq i \leq p$.

Then $F_{(\pi_1,\pi_2,...,\pi_p)}$ is also an edge-graceful labeling of G.

Theorem 2. Let G be a edge-graceful connected (p, q) graph with $q = kp + r$, where $0 \leq r < p$, $k \geq 1$, k and r are integers. Then every edge-graceful labeling of G induces $[(k+1)!]^r[k!]^{p-r}$ distinct edge-graceful labelings of G.

[The above Theorems 1 and 2 are publishesd in Electronic Notes in Discrete Mathematics, Elsevier, 53(2016), 287-296.]

Theorem 3. Let G be a k-edge-graceful connected (p, q) graph with $q = k'p + r$, where $0 \leq r < p$, $k' \geq 1$, k' and r are integers. Then every k-edge-graceful labeling of G induces $[(k'+1)!^r][k'!]^{p-r}$ distinct k-edge-graceful labelings of G.

In Chapter 3, we enumerate edge-graceful labelings of K_3, K_4, K_5.

Theorem 1. K_3 has unique edge-graceful labeling and K_4 has 8 edge-graceful labelings.

Theorem 2. K_5 has 6560 edge-graceful labelings.

[The above Theorems 1 and 2 are publishesd in Mathematical

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In Chapter 4, we derive new directed edge-graceful labelings from a given one. Since the directed edge-graceful labeling is defined with respect to an orientation, the new directed edge-graceful labelings are induced by both the labelings and orientations. Hence more number of directed edge-graceful labelings are derived than that of edge-graceful labelings.

Given a directed edge-graceful labeling $f_{D(G)}$ of G, $f'_{D(G)}(v)$ is defined with respect to modulo p. Since $q > p$, some arcs have labels in the same congruence class of modulo p. We use this idea to derive new labelings by permuting the arc labels in the same congruence class.

The following are our main results.

Theorem 1. Let $f_{D(G)}$ be a directed edge-graceful labeling of a graph G of order p, size $q = kp + r$, where k and r are integers with $k \geq 1$, $0 \leq r < p$. For $1 \leq j \leq r$, let π_j denote a permutation on the set $\{0, 1, 2, ..., k\}$; and for $r + 1 \leq j \leq p$, let π_j denote a permutation on the set $\{0, 1, 2, ..., k - 1\}$. Let $F_{D(G)(\pi_1,\pi_2,...,\pi_p)}$ be a labeling, defined as,

$F_{D(G)(\pi_1,\pi_2,...,\pi_p)}(a) = \pi_i(l)p + i$ if $f_{D(G)}(a) = lp + i$,

where $0 \leq l \leq k$ if $1 \leq i \leq r$; and $0 \leq l \leq k - 1$ if $r + 1 \leq i \leq p$.

Then $F_{D(G)(\pi_1,\pi_2,...,\pi_p)}$ is also a directed edge-graceful labeling of G.

Theorem 2. Let $f_{D(G)}$ be a directed edge-graceful labeling of a graph of order p and size $q = kp + r$, where $k \geq 1$ and $0 \leq r < p$. Let $\mathscr{A}_{f_{D(G)}}$ denote the set of all directed edge-graceful labelings defined in Theorem 1, induced by $f_{D(G)}$. Then $|\mathscr{A}_{f_{D(G)}}| = [(k + 1)!]^r (k!)^{p-r}$ and all the labelings in $\mathscr{A}_{f_{D(G)}}$ are non-isomorphic.

Definition: Given an orientation $D(G)$, the *converse orientation* of $D(G)$ is obtained from $D(G)$ by reversing the direction of every arc of $D(G)$.

The converse orientation of $D(G)$ is denoted by $D'(G)$.

Theorem 3. Let $f_{D(G)}$ be a directed edge-graceful labeling of a graph G of order p. Define $f_{D'(G)}(\overrightarrow{vu}) = f_{D(G)}(\overrightarrow{uv})$, $\forall\ \overrightarrow{uv} \in A(D(G))$. Then $f_{D'(G)}$ is also a directed edge-graceful labeling of G.

Next, we induce new orientations, by changing the directions of the arcs, whose labels are multiple of p.

Given a directed edge-graceful labeling $f_{D(G)}$ of G, let
$A_p(D(G)) = \{a \in A(D(G)) \mid f_{D(G)}(a) \equiv 0 \pmod{p} \}$.
Since $f'_{D(G)}(v) = f^{+}_{D(G)}(v) - f^{-}_{D(G)}(v) \pmod{p}$, by *fixing the arc labels*,
but *changing the direction of any arc in* A_p, we can derive new directed edge-graceful labelings with new orientations.
Given a directed edge-graceful labeling $f_{D(G)}$ of a graph G of order p and size $q = kp + r$, where $k \geq 1$, $0 \leq r < p$, let $\mathscr{D}_{f_{D(G)}}$ denote the collection of orientations $D_1(G)$ (derived from $f_{D(G)}$), satisfying the following conditions:

(i) For every $\overrightarrow{uv} \in A \cap A_p$, either $\overrightarrow{uv} \in A(D_1(G))$ or $\overrightarrow{vu} \in A(D_1(G))$.

(ii) For every $\overrightarrow{uv} \in A - A_p$, $\overrightarrow{uv} \in A(D_1(G))$.

Theorem 4. Let $f_{D(G)}$ be a directed edge-graceful labeling of a graph of order p and size $q(> p)$. Let $D_1(G) \in \mathscr{D}_{f_{D(G)}}$.
For any $\overrightarrow{uv} \in A(D_1(G))$, define

$$f_{D_1(G)}(\overrightarrow{uv}) = \begin{cases} f_{D(G)}(\overrightarrow{uv}) & \text{if } \overrightarrow{uv} \in A(D(G)) \\ f_{D(G)}(\overrightarrow{vu}) & \text{if } \overrightarrow{vu} \in A(D(G)). \end{cases}$$

Then $f_{D_1(G)}$ is also a directed edge-graceful labeling of G.

Theorem 5. Let $f_{D(G)}$ be a directed edge-graceful labeling of a graph of order p and size $q > p$. Let $D_1(G) \in \mathscr{D}_{f_{D'(G)}}(G)$.
For any $\overrightarrow{uv} \in A(D_1(G))$, define

$$f_{D_1(G)}(\overrightarrow{uv}) = \begin{cases} f_{D'(G)}(\overrightarrow{uv}) & \text{if } \overrightarrow{uv} \in A(D'(G)) \\ f_{D'(G)}(\overrightarrow{vu}) & \text{if } \overrightarrow{vu} \in A(D'(G)). \end{cases}$$

Then $f_{D_1(G)}$ is also a directed edge-graceful labeling of G.
In the next results, for our convenience, we represent the *arcs using the labels of their tail and head.*

i.e. given a directed edge-graceful labeling $f_{D(G)}$ of G,
$\overrightarrow{uv}$ in $A(D(G))$ is denoted by $(f'_{D(G)}(u), f'_{D(G)}(v))$.

Theorem 6. Let $f_{D(G)}$ be a directed edge-graceful labeling of G of order p

and size $q = kp + r$, where $k \geq 1$ and $0 \leq r < p$. Suppose that, for every arc $(\alpha, \beta) \in A(D(G))$, exactly one of the following conditions hold:

(i) $(p - \beta, p - \alpha) \in A(D(G))$ and $f_{D(G)}(\alpha, \beta) \equiv f_{D(G)}((p - \beta, p - \alpha)) \pmod p$

(ii)$(p-\alpha, p-\beta) \in A(D(G))andf_{D(G)}(\alpha, \beta) \equiv f_{D(G)}((p-\alpha, p-\beta)) \equiv 0(\bmod p)$.

Then $f_{D(G)}$ induces $2^k[(k + 1)!]^r[k!]^{p-r}$ non-isomorphic directed edge-graceful labelings of G.

Theorem 7. Let $f_{D(G)}$ be a directed edge-graceful labeling of G of order p and size $q = kp + r$,where $k \geq 1$ and $0 \leq r < p$. Suppose that there exists an arc $(\alpha, \beta) \in A(D(G))$ satisfying one of the following conditions:

(i) $(p - \beta, p - \alpha), (p - \alpha, p - \beta) \notin A(D(G))$

(ii) $(p - \beta, p - \alpha) \in A(D(G))$, $f_{D(G)}((\alpha, \beta)) \not\equiv f_{D(G)}((p - \beta, p - \alpha)) \pmod p$

(iii) $(p - \alpha, p - \beta) \in A(D(G))$, $f_{D(G)}((\alpha, \beta)) \not\equiv f_{D(G)}((p - \alpha, p - \beta)) \pmod p$.

Then $f_{D(G)}$ induces $2^{k+1}((k+1)!)^r(k!)^{p-r}$ non-isomorphic directed edge-graceful labelings of G.

In Chapter 5, from a given directed edge-graceful graph G of order p and size q, we construct new directed edge-graceful graphs by adding an arc with label $q + 1$ or by removing an arc with label q.

Theorem 1. Let $G(\neq K_p)$ be a directed edge-graceful graph of order p and size q with a labeling $f_{D(G)}$. 1. If $q + 1 \equiv 0 \pmod p$, then $G + a$ is also a directed edge-graceful graph, for every $a \notin D(G)$.

2. Let $q + 1 \equiv s \pmod p$, where $s \neq 0$. If there exist α, β such that $\alpha - \beta \equiv s \pmod p$ and $(\alpha, \beta) \notin D(G)$, then $G + (\alpha, \beta)$ is also a directed edge-graceful graph.

Theorem 2. Let G be a directed edge-graceful graph of order p and size q with a labeling $f_{D(G)}$. Let $q \equiv s$ (mod p). If $(\alpha, \beta) \in D(G)$ with $f_{D(G)}(\alpha, \beta) = q$ and $\beta - \alpha \equiv s$ (mod p), then $G - (\alpha, \beta)$ is also a directed edge-graceful graph.

We also construct new directed edge-graceful graphs from K_n by adding two vertices and two arcs. Let $f_{D(K_n)}$ be a directed edge-graceful labeling of K_n. By adding two pendent arcs and two vertices to K_n, we shall construct a new directed edge-graceful $(n+2, \binom{n}{2}+2)$ graph K_n^* with a directed edge-graceful labeling $f_{D(K_n^*)}$, where the orientation $D(K_n^*)$ of K_n^* is an extension of $D(K_n)$ and $f_{D(K_n^*)}|_{D(K_n)}$ is $f_{D(K_n)}$.

As $f'_{D(K_n^*)}$ is to be calculated with respect to modulo $(n+2)$, we shall first determine the labels of the vertices of K_n with respect to modulo $(n+2)$.

For any $u \in V(K_n)$,

$l(u) = (\sum_{x \in N^+_{D(K_n)}(u)} f_{D(K_n)}(x) - \sum_{y \in N^-_{D(K_n)}(u)} f_{D(K_n)}(y)) \pmod{n+2}$,

$L = \{l(u) \mid u \in V(K_n)\}$ and $L^c = \{0, 1, 2, ..., n+1\} - L$.

For our convenience, in the following theorems, we set

$\alpha_1^- \equiv \alpha - 4 \pmod{n+2}$,

$\alpha_2^- \equiv \alpha - 5 \pmod{n+2}$,

$\alpha_1^+ \equiv \alpha + 4 \pmod{n+2}$,

and $\alpha_2^+ \equiv \alpha + 5 \pmod{n+2}$,

where $\alpha, \alpha_1^-, \alpha_2^-, \alpha_1^+, \alpha_2^+ \in \{0, 1, 2, ..., n+1\}$.

Theorem 3. Let $f_{D(K_n)}$ be a directed edge-graceful labeling of K_n with $|L| = n-2$ and let v_1, v_1', v_2, v_2' be the vertices with $l(v_1) = l(v_1') = \alpha$, $l(v_2) = l(v_2') = \beta$. If $L^c \in \{\{\alpha_1^+, \beta_2^+, n-2, n-3\}, \{\alpha_1^-, \beta_2^-, 4, 5\}$, $\{\alpha_1^+, \beta_2^-, n-2, 5\}, \{\alpha_1^-, \beta_2^+, 4, n-3\}, \{\alpha_2^+, \beta_1^+, n-2, n-3\}, \{\alpha_2^+, \beta_1^-, 4, n-3\}$, $\{\alpha_2^-, \beta_1^+, n-2, 5\}, \{\alpha_2^-, \beta_1^-, 4, 5\}\}$, then there exists an orientation $D(K_n^*)$ of K_n^* and a labeling $f_{D(K_n^*)}$, such that K_n^* is a directed edge-graceful graph.

Theorem 4. Let $f_{D(K_n)}$ be a directed edge-graceful labeling of K_n with $|L| = n-2$ and let v_1, v_2, v_3 be the vertices of K_n having the same l-value α. If $L^c \in \{\{\alpha_1^+, \alpha_2^+, n-2, n-3\}, \{\alpha_1^-, \alpha_2^-, 4, 5\}, \{\alpha_1^+, \alpha_2^-, n-2, 5\}$, $\{\alpha_1^-, \alpha_2^+, 4, n-3\}\}$, then there exists an orientation $D(K_n^*)$ of K_n^* and a labeling $f_{D(K_n^*)}$, such that K_n^* is a directed edge-graceful graph.

Theorem 5. Let $f_{D(K_n)}$ be a directed edge-graceful labeling of K_n, with $|L| = n-1$ and let v_1, v_1' be the vertices of K_n having the same l-value α.

If (i) $L^c \in \{\{4, 5, \alpha_1^-, \alpha_2^-\} - \{\alpha\}, \{5, n-2, \alpha_1^+, \alpha_2^-\} - \{\alpha\}$,
$\{4, n-3, \alpha_1^-, \alpha_2^+\} - \{\alpha\}, \{n-2, n-3, \alpha_1^+, \alpha_2^+\} - \{\alpha\}, \{4, 5, \alpha - 9\}$,
$\{5, n+1, \alpha_1^-\}, \{0, 4, \alpha_1^-\}, \{1, 4, \alpha_2^-\}, \{0, 5, \alpha_2^-\}, \{4, n-3, \alpha+1\}, \{9, n-3, \alpha_1^-\}$,
$\{5, n-2, \alpha-1\}, \{9, n-2, \alpha_2^-\}, \{4, n-7, \alpha_2^+\}, \{0, n-3, \alpha_2^+\}, \{5, n-7, \alpha_1^+\}$,

$\{0, n-2, \alpha_1^+\}, \{1, n-3, \alpha_1^+\}, \{n-2, n-3, \alpha+9\}, \{n-2, n+1, \alpha_2^+\}\}$, then there exists an orientation $D(K_n^*)$ of K_n^* and a labeling $f_{D(K_n^*)}$ such that K_n^* is a directed edge-graceful graph.

Theorem 6. Let $f_{D(K_n)}$ be a directed edge-graceful labeling of K_n with $|L| = n-1$ and $v_1, v_1^{'}$ be the vertices of K_n such that $l(v_1) = l(v_1^{'}) = \alpha$.
If $L^c \in \{\{5, n+1, \alpha_1^-\}, \{1, 4, \alpha_2^-\}, \{1, n-3, \alpha_1^+\}, \{n-2, n+1, \alpha_2^+\},$
$\{5, n-7, \alpha_1^+\}, \{4, n-7, \alpha_2^+\}, \{9, n-3, \alpha_1^-\}, \{9, n-2, \alpha_2^-\}$, then there exists an orientation $D(K_n^*)$ of K_n^* and a labeling $f_{D(K_n^*)}$ such that K_n^* is a directed edge-graceful graph.

Theorem 7. Let $f_{D(K_n)}$ be a directed edge-graceful labeling of K_n with $|L| = n-1$ and $v_1, v_1^{'}$ be the vertices of K_n such that $l(v_1) = l(v_1^{'}) = \alpha$. If $L^c = \{\{4, 5, \alpha-9\}, \{n-2, n-3, \alpha+9\}, \{5, n-2, \alpha-1\}, \{4, n-3, \alpha+1\}\}$, then there exists an orientation $D(K_n^*)$ of K_n^* and labeling $f_{D(K_n^*)}$ such that K_n^* is a directed edge-graceful graph.

Theorem 8. Let $f_{D(K_n^*)}$ be a directed edge-graceful labeling of K_n with $|L| = n$. If there exists $\alpha, \beta \in L$ such that $L^c \cup \{\alpha, \beta\}$. If
$L^{'} \in \{\{4, 5, \alpha_1^-, \beta_2^-\}, \{n-2, n-3, \alpha_1^+, \beta_2^+\}, \{5, n-2, \alpha_1^+, \beta_2^-\}, \{4, n-3, \alpha_1^-, \beta_2^+\}\}$, then there exists an orientation $D(K_n^*)$ of K_n^* and labeling $f_{D(K_n^*)}$ such that K_n^* is a directed edge-graceful graph.

Theorem 9. Let $f_{D(K_n^*)}$ be a directed edge-graceful labeling of K_n with $|L| = n$. If $L^c \in \{\{5, n-3\}, \{1, n+1\}, \{4, n-2\}, \{9, n-7\}, \{4, n-2\}\}$, then there exists an orientation $D(K_n^*)$ of K_n^* and labeling $f_{D(K_n^*)}$ such that K_n^* is a directed edge-graceful graph.

Theorem 10. Let $f_{D(K_n^*)}$ be a directed edge-graceful labeling of K_n with $|L| = n$. If $L^c \in \{\{5, n-3\}, \{4, n-2\}\}$, then there exists an orientation $D(K_n^*)$ of K_n^* and a labeling $f_{D(K_n^*)}$ such that K_n^* is a directed edge-graceful graph.

In Chapter 6, we define new labelings called *arc-graceful labeling* and *k- arc-graceful labeling* for *digraphs* and enumerate the new labelings derived from the existing one.

Arc-graceful labeling: A digraph $D=(V,A)$ with $|V|=p$, $|A|=q$, $d^{+}(v)>0$ and $d^{-}(v)>0$, for all $v\in V$, is said to be an *arc-graceful graph* if there exists a bijection $f:A\to\{1,2,...,q\}$ such that the induced mappings
(i) $f':V\to\{0,1,2,...,p-1\}$ defined by $f'(v_i)=\sum\limits_{a\in N^{-}(v)} f(a)$ (mod p) and
(ii) $f'':V\to\{0,1,2,...,p-1\}$ defined by $f''(v_i)=\sum\limits_{a\in N^{+}(v)} f(a)$ (mod p)
are bijective.

If D is an arc-graceful graph with respect to a labeling f, then f is called an *arc-graceful labeling* of D.

k-arc-graceful labeling: A digraph D with $d^{+}(v)>0$, $d^{-}(v)>0$, for all $v\in V$, is said to be an *k-arc-graceful graph* if there exists a bijection
$f:A\to\{k,k+1,k+2,...,k+q-1\}$ such that the induced mappings
(i) $f^{'}:V\to\{0,1,2,...,p-1\}$ defined by $f'(v_i)=\sum\limits_{a\in N^{-}(v)} f(a)$ (mod p) and
(ii) $f^{''}:V\to\{0,1,2,...,p-1\}$ defined by $f''(v_i)=\sum\limits_{a\in N^{+}(v)} f(a)$ (mod p)
are bijective.

If D is an arc-graceful graph with respect to a labeling f, then the labeling f is called a *k-arc-graceful labeling* of D.

Our main results are the following theorems.

Theorem 1. If a digraph D of order p and size q is arc-graceful, then

(i) $q(q+1)\equiv 0 \pmod{p}$

and (ii) $\frac{q(q+1)}{2}\equiv 0 \pmod{p}$, if p is odd.

Theorem 2. Let D be an arc-graceful digraph of order p and size $q(>p)$ with $q=kp+r$, where $k\geq 1$, $0\leq r<p$. Then every arc-graceful labeling f of D induces $[(k+1)!^{r}][k!]^{p-r}$ distinct arc-graceful labelings for D.

Theorem 3. Let G be a k-edge-graceful connected (p,q) graph with $q=k'p+r$, where $0\leq r<p$, $k'\geq 1$, k' and r are integers. Then every k-edge-graceful labeling of G induces $[(k'+1)!^{r}][k'!]^{p-r}$ distinct k-edge-graceful labelings of G.

Conclusion

In this thesis, we derive new labelings from the existing labeling of graphs and digraphs.

We derive new edge-graceful labelings of a graph from an existing one and also enumerate them. A similar work is done for k-edge-graceful labelings also.

We have enumerated all edge-graceful labelings of K_3, K_4, K_5.

Given a directed edge-graceful labeling $f_{D(G)}$ of G, we derive many new orientations from $f_{D(G)}$ and using these orientations, we derive more number of directed edge-graceful labelings. Also we enumerate them.

We not only construct new labelings from an existing one, but also we construct new directed edge-graceful graphs from a given directed edge-graceful graph G, by adding or deleting an arc from G. Also we construct new directed edge-graceful graphs by adding two new vertices and two new arcs to K_n, using the directed edge-graceful labeling of K_n, under three types of construction.

Also, we define new labelings called arc-graceful labeling and k-arc-graceful labeling for digraphs and we enumerate the new labelings derived from the existing one.

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