Synopsis of the Ph.D. Thesis entitled

Magic Labelings of graphs

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Synopsis
Magic Labelings of Graphs

Graph Theory is an important field of Mathematics and Computer Science. Graphs are useful Mathematical tools for modeling the relationships among objects, which are represented by vertices. In their turn, relationships between vertices are represented by edges. In this context, graph theory received considerable attention not only from the mathematical community, but also from the whole scientific community.

One of the main emerging concepts in Graph Theory is Magic labelings of graphs and in particular Edge-magic labelings of graphs. Let G be a graph with p vertices and q edges. An *edge-magic total labeling(EMT labeling)* f is a bijection from $V(G) \cup E(G)$ to the set of integers $\{1,\ 2, ..., p+q\}$ such that if xy is an edge of G, then $f(x) + f(y) + f(xy) = \lambda$ for some integer constant λ. A graph with an edge-magic total labeling is called an EMT graph.

Edge-magic total labeling is applied in communication networks [3]. Suppose that it is necessary to assign addresses to the possible links in a communications network. It is required that the addresses all be different, and that the address of a link to be deduced from the identities of the two nodes linked, without having to use a lookup table.

First, a graph is constructed with the nodes as vertices and edges between all pairs of nodes, where a link is provided. The vertices are labeled in such a way that the differences between endpoints of edges are all distinct - this is called a *semi-graceful* labeling. Then the address of a link is the difference between the labels on its endpoints.

The following solution, using an edge-magic total labeling, provides some additional information[10]. Suppose that the graph has an edge-magic total labeling λ with magic constant k. The nodes and links are assigned the labels of the corresponding vertices and edges. Then the address of the link from x to y is readily calculated as $k - \lambda(x) - \lambda(y)$. Moreover, a unique address is available for messages from the system operator to the nodes: the link from the system operator to node x receives address $\lambda(x)$.

In 1920, Kotzig and Rosa [9] defined a *magic valuation* of a graph $G(V,\ E)$ as a bijection f from $V \cup E$ to $\{1,\ 2,\ \cdots,\ |V \cup E|\}$ such that for all edges xy, $f(x)+f(y)+f(xy)$ is a constant (called the *magic constant*). This notion was rediscovered by Ringel and Llado [12] in 1996, who called this labeling edge-magic. Enomoto, Llado, Nakamigawa and Ringel [5] called a graph $G(V,\ E)$ with an edge-magic total labeling (EMT labeling) that has the additional property that the vertex labels are 1 to $|V|$, a $super\ edge-magic\ graph$, and the labeling $super\ edge-magic\ total\ labeling\ (SEMT\ lableing)$. In 2001, Figueroa et al.[6] exhibited the relationships between super edge-magic labelings and other well-studied classes of labelings.

Ming Yao et al. [11] constructed several SEMT graphs of larger order from other SEMT graphs of small orders. Baskoro, Sudarsana and Cholily [2] provided some constructions of new SEMT graphs from some old ones by attaching 1, 2 or 3 pendant vertices and edges. In [8], Kim introduced a new construction of new SEMT graphs by attaching odd number of pendant vertices and edges under some conditions. For more results on edge-magic total labelings and super edge-magic total labelings, one can refer [7].

Graph labelings were first introduced in the late 1960"s. Since then many varieties of graph labelings have been investigated [7]. Despite the unabated procession of papers, there are few results on the enumeration of labelings of graphs. Also there are only few results on construction of new graphs from the old ones. This motivates us to study the structure of SEMT graphs and enumerate the SEMT labelings of these graphs, and also to search for some methods to construct new SEMT graphs from the old ones.

In our thesis, we have enumerated SEMT labelings and EMT labelings of connected unicyclic (p,q) graphs with $\lambda = p + q + 3$. Also we deal with construction of connected, unicyclic SEMT graphs and EMT graphs from an existing one.

Summary of our Results

We plan to present our results in four chapters.

Connected unicyclic graphs with magic constant $p+q+3$

In Chapter 2, we first study the structure of connected unicyclic SEMT graphs with magic constant $\lambda = p+q+3$. Next, using this structure, we enumerate SEMT labelings with $\lambda = p+q+3$, for connected unicyclic graphs with $\Delta = p-1,\ p-2,\ p-3$ and $p-4$.

Let $\mathcal{G}$ denote the class of all connected unicyclic SEMT graphs with magic constant $\lambda = p+q+3$. Let $\mathcal{G}_{\mathbf{m}}$ denote the subclass of $\mathcal{G}$, containing the graphs with $\Delta = m$.

In $\mathcal{G}_{p-1}$, $K_{1,\ p-1}+e$ is the only graph.

Theorem 1. Let G be any graph in $\mathcal{G}$. Then G contains K_3 (and so $K_3 = C_3$ is the only cycle in G). Moreover, in any SEMT labeling with $\lambda = p+q+3$, the vertex v_1 lies in K_3.

Theorem 2. The graph $K_{1,\ p-1}+e$ with $p \geq 4$ has $\left\lfloor \frac{p+1}{2} \right\rfloor$ SEMT labelings.

Theorem 3. When $p \geq 6$, the total number of SEMT labelings of graphs in $\mathcal{G}_{p-2}$ is

$$S = \begin{cases} \frac{(p-3)(p-2)+6}{4} & \text{if } p \equiv 0 \ (mod\ 4) \\ \frac{(p-3)(p-2)+4}{4} & \text{if } p \equiv 2 \ (mod\ 4) \\ \frac{(p-3)(p-1)+4}{4} & \text{if } p \equiv 1 \ or\ 3 \ (mod\ 4). \end{cases}$$

Theorem 4. When $p \geq 8$, the number of SEMT labelings of graphs in $\mathcal{G}_{p-3}$ is

$$S = \begin{cases} \frac{2p^3-18p^2+85p-168}{24} & \text{if } p \equiv 0 \ (mod\ 8) \\ \frac{4p^3-33p^2+139p-201}{48} & \text{if } p \equiv 1 \ or\ 5 \ (mod\ 8) \\ \frac{2p^3-18p^2+82p-132}{24} & \text{if } p \equiv 2 \ or\ 6 \ (mod\ 8) \\ \frac{4p^3-33p^2+134p-213}{48} & \text{if } p \equiv 3 \ or\ 7 \ (mod\ 8) \\ \frac{2p^3-18p^2+85p-156}{24} & \text{if } p \equiv 4 \ (mod\ 8). \end{cases}$$

Theorem 5. When $p \geq 10$, the number of SEMT labelings of graphs in $\mathcal{G}_{p-4}$ is

$$S = \begin{cases} \frac{4p^4-56p^3+317p^2-544p-576}{192} & \text{if } p \equiv 0 \ (mod\ 16) \ \text{ and when } p < 24 \\ & (i.e.\ when\ p = 16) \\ \frac{4p^4-56p^3+317p^2-556p-384}{192} & \text{if } p \equiv 0 \ (mod\ 16) \ \text{ and when } p \geq 24 \\ \frac{4p^4-56p^3+317p^2-556p-480}{192} & \text{if } p \equiv 8 \ (mod\ 16) \\ \frac{4p^4-56p^3+302p^2-340p-1446}{192} & \text{if } p \equiv 1 \ (mod\ 8)) \\ \frac{4p^4-56p^3+314p^2-472p-1080}{192} & \text{if } p \equiv 2 \ (mod\ 8) \\ \frac{4p^4-56p^3+302p^2-364p-1206}{192} & \text{if } p \equiv 3 \ (mod\ 8) \\ \frac{4p^4-56p^3+317p^2-544p-720}{192} & \text{if } p \equiv 4 \ (mod\ 8) \\ \frac{4p^4-56p^3+302p^2-340p-1350}{192} & \text{if } p \equiv 5 \ (mod\ 8) \\ \frac{4p^4-56p^3+314p^2-472p-984}{192} & \text{if } p \equiv 6 \ (mod\ 8) \\ \frac{4p^4-56p^3+302p^2-364p-1110}{192} & \text{if } p \equiv 7 \ (mod\ 8). \end{cases}$$

Chapter 3 : EMT labelings and SEMT labelings of graphs in $\mathcal{G}_{\mathbf{p-1}}$, $\mathcal{G}_{\mathbf{p-2}}$ and $\mathcal{G}_{\mathbf{p-3}}$

Given a SEMT labeling of G, by interchanging the label of a pendant vertex of G with the label of its incident edge, we get a new EMT labeling with the same magic constant. In Chapter 3, we enumerate EMT labelings of graphs in $\mathcal{G}_{p-1}$, $\mathcal{G}_{p-2}$ and $\mathcal{G}_{p-3}$, where $\mathcal{G}_m$ denotes the class of all connected unicyclic SEMT graphs with the magic constant $p+q+3$ and $\Delta = m$. We use the structure of these graphs, which are dealt in Chapter 2, for the enumeration. Using duality, we enumerate the SEMT labelings and EMT labelings wth the magic constant $\lambda = 3p$ of these graphs.

Theorem 1. Let G be a SEMT graph with magic constant λ and let G have k pendant vertices. Then every SEMT labeling of G induces 2^k EMT labelings with the same magic constant λ.

Theorem 2. If a graph G has n SEMT labelings and k pendant vertices, then G has $2^k n$ EMT labelings.

Theorem 3. The graph $K_{1,\ p-1} + e$ with $p \geq 4$, has $2^{p-3} \left\lfloor \frac{p+1}{2} \right\rfloor$ EMT labelings with $\lambda = p + q + 3$.

Theorem 4. When $p \geq 6$, the total number of EMT labelings of graphs in $\mathcal{G}_{p-2}$ is

$$\begin{cases} (\frac{p-2}{2})2^{p-3} + (\frac{(p^2-7p+16}{4})2^{p-4} & \text{if } p \equiv 0\ (mod\ 12) \\ (\frac{p+1}{2})2^{p-3} + (\frac{(p-5)(p-1)}{4})2^{p-4} & \text{if } p \equiv 1, 5, 7, 11\ (mod\ 12) \\ (\frac{p}{2})2^{p-3} + (\frac{(p-5)(p-2)}{4})2^{p-4} & \text{if } p \equiv 2, 10\ (mod\ 12) \\ (\frac{p-1}{2})2^{p-3} + (\frac{(p-3)^2}{4})2^{p-4} & \text{if } p \equiv 3, 9\ (mod\ 12) \\ (\frac{p}{2})2^{p-3} + (\frac{(p-4)(p-3)}{4})2^{p-4} & \text{if } p \equiv 4, 8\ (mod\ 12) \\ (\frac{p-2}{2})2^{p-3} + (\frac{p-5)(p-2)}{4} + 1)2^{p-4} & \text{if } p \equiv 6\ (mod\ 12). \end{cases}$$

Applying duality to all the above theorems we get the same number of EMT and SEMT labelings of graphs with $\lambda = 3p$ as that of graphs with $\lambda = p + q + 3$.

Chapter 4 : Construction of new SEMT $(\mathbf{p+n},\ \mathbf{q+n})$ graphs from a SEMT $(\mathbf{p},\ \mathbf{q})$ graph with $\mathbf{n \leq p+1}$

In this chapter, we introduce the notation $B_{n,\ s_1,\ s_2}$ to denote the board of size n (where $s_1+s_2 \leq n$), having the darkened squares only in the following $(i,\ j)^{th}$ cells in the first s_1 rows and in the last s_2 rows: (i) $1 \leq i \leq s_1$ and $1 \leq j \leq s_1 - i + 1$ (ii) $n - s_2 + 1 \leq i \leq n$ and $s_2 - n + i \leq j \leq n$.

Let G be a SEMT $(p,\ q)$ graph with the magic constant λ, and let f be a corresponding SEMT labeling. Let $V(G) = \{v_1, v_2, \cdots, v_p\}$ and $E(G) = \{e_1, e_2, \cdots, e_q\}$. Without loss of generality, let $f(v_i) = i$ for $i = 1, 2, \cdots, p$, and $f(e_j) = p + j,\ 1 \leq j \leq q$. We construct new SEMT (p + n, q + n) graphs with the magic constant $\lambda + 2n$, by adding n pendant edges (and n pendant vertices) to G. Next, we represent these possible incidences between the new vertices $v_{p+1},\ v_{p+2}, \cdots, v_{p+n}$ and the new edges $e_{q+1},\ \ e_{q+2}, \cdots, e_{q+n}$ by an $n \times n$ array of squares, where the rows correspond to the vertices and the columns correspond to the edges. The forbidden incidences are denoted by the darkened squares. Then the corresponding board is $B_{n,\ s_1,\ s_2}$. Then **the number of possible incidences between these new vertices and edges would be equal to the number of new SEMT $(\mathbf{p+n},\ \mathbf{q+n})$ graphs**.

We count this number using combinatorial analysis.

We define $r_k(B)$ to be the number of ways of choosing k darkened squares from the board B, each in a different row and column.

Theorem 1. Let G be a connected SEMT (p, q) graph with the magic constant λ. If $n \leq p+1$, there are T connected SEMT (p + n, q + n) graphs with the magic constant $\lambda + 2n$, where

$$T = n! - r_1(B)(n-1)! + r_2(B)(n-2)! - r_3(B)(n-3)! + r_4(B)(n-4)! - \cdots$$
$$\cdots + (-1)^{s_1+s_2} r_{s_1+s_2}(B)(n - s_1 - s_2)!,$$

$s_1 = \lambda + n - 3p - 2,\ s_2 = n - \lambda + 2p + 1$ and $B = B_{n,\ s_1,\ s_2}$.

Chapter 5 : Construction of new SEMT $(\mathbf{p+n},\ \mathbf{q+n})$ graphs from a SEMT $(\mathbf{p},\ \mathbf{q})$ graph with $\mathbf{n=p+2}$

In this chapter, we enumerate new SEMT $(p+n,\ q+n)$ graphs when $n=p+2$.

Theorem 1. Let G be a connected SEMT (p, q) graph with magic constant λ. If $n=p+2$, then there are T connected SEMT (p + n, q + n) graphs, where

$$T = n! - r_1(B)(n-1)! + r_2(B)(n-2)! - r_3(B)(n-3)! + r_4(B)(n-4)! - \cdots$$
$$\cdots + (-1)^{s_1+s_2-1} r_{s_1+s_2-1}(B)(n-s_1-s_2+1)!,$$
$$s_1 = \lambda + n - 3p - 2,\ s_2 = n - \lambda + 2p + 1 \text{ and } B = B_{n,\ s_1,\ s_2-1}.$$

Conclusion

In this thesis we have studied the structure of graphs in $\mathcal{G}_m$ with $\Delta = p-1,\ p-2,\ p-3\ and\ p-4$, where $\mathcal{G}_m$ denotes the class of all connected unicyclic SEMT graphs with magic constant $\lambda = p+q+3$. We have enumerated the SEMT labelings of these graphs.

We have also enumerated the EMT labelings these graphs. Using duality we have also enumerated the SEMT and EMT labelings of graphs in $\mathcal{G}_m$ with $\Delta = p-1,\ p-2\ and\ p-3$ with the magic constant $\lambda = 3p$.

We have constructed new SEMT $(p+n,\ q+n)$ graphs with the magic constant $\lambda + 2n$ from a given SEMT $(p,\ q)$ graph with the magic constant λ, when $n < p+2$ and $n = p+2$. We have also enumerated these newly constructed SEMT graphs.

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